**Indian Statistical Institute, Bangalore**

B.Math (Hons.) III Year, First Semester

Semestral Examination

Complex Analysis

Time: 3 $\frac{1}{2}$ hours — 2 December 2011 — Instructor: B.Bagchi

Maximum marks: 100

Note: Each Question carries 20 Marks.

1. (a) State and prove Schwarz's Lemma.

   (b) Let $U$ be a subdomain of the unit disc $\mathbb{D}$; $0 \,\varepsilon\, U$. Suppose $f : U \to \mathbb{D}$ is a biholomorphic map such that $f(0) = 0$. Then show that $\mid f'(0) \mid \geq 1$ and $\mid f(z) \mid \geq \mid z \mid$ for all $z \,\varepsilon\, U$.

   (c) If $\Omega$ is a simply connected proper subdomain of $\mathbb{C}$ then show that there is a one-one holomorphic function from $\Omega$ into $\mathbb{D}$.

   (No! you may not assume Riemann mapping theorem!)

2. (a) Let $f$ be an entire function such that $\lim\limits_{z\to\infty} f(z) = \infty$.Then show that $f$ is a polynomial.

   (b) If $f$ is an one-one entire function then show that $f$ satisfies the hypothesis of part (a). Hence deduce that $f(z) \equiv az + b$ for some $a \,\varepsilon\, \mathbb{C}^*$, $b \,\varepsilon\, \mathbb{C}$.

3. (a) If $f : \Omega \to \mathbb{C}$ is a one-one holomorphic function then show that $f'(z) \neq 0$ for all $z\varepsilon\Omega$.

   (b) if $f : \Omega \to \mathbb{C}$ is a holomorphic function such that $f'(z) \neq 0$ for all $z\varepsilon\Omega$ then show that $f$ is locally one-one.

4. Let $\Omega$ be a bounded planar domain and let $f : \overline{\Omega} \to \mathbb{C}$ be a continuous function which is non-constant and holomorphic on $\Omega$.

   (a) Show that $\partial f(\Omega) \subseteq f(\partial\Omega)$.

   (b) If $U$ is a planar domain such that $U \cap f(\Omega) \neq \varphi$ but $U \cap f(\partial\Omega) = \varphi$ then show that $U \subseteq f(\Omega)$.

5. Let $\Omega$ be a simply connected domain and let $\mathcal{F} = \{f\varepsilon H(\Omega) : f(z) \neq \pm 1$ for all $z\varepsilon\Omega\}$.

   (a) Show that, for each $f\varepsilon\, \mathcal{F}$, there is a $g\varepsilon H(\Omega)$ such that $f(z) \equiv cos(\pi g(z))$.

   (b) Hence show that there is an absolute constant $r_o > 0$ such that for any $f\varepsilon\, \mathcal{F}$, $g(\Omega)$ does not contain any disc of radius $r_o$. Find an explicit value of $r_o$.